INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 39, Northern Spring 2018 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An angle bisector and an altitude emanating from the same vertex of a triangle divide the opposite side into three parts. Is it possible that a new triangle may be constructed from those three parts? (3 points)

2. Four positive integers are given such that each of them is divisible by the greatest common divisor of the other three numbers, and the least common multiple of any three is divisible by the fourth number. Prove that the product of these four numbers is a perfect square. (4 points)

3. Two circles Γ_1 and Γ_2, with centres O_1 and O_2 respectively, touch externally at point T. A common tangent touches Γ_1 at point A and Γ_2 at point B. A common tangent to both circles at point T meets the line AB at point M. Suppose AC is a diameter of Γ_1. Prove that CM and AO_2 are perpendicular to each other. (4 points)

4. There is a checker in the corner square of an 8×8 chessboard. Petya and Vasya take turns moving the checker. Petya starts first, and on his turn he moves as a chess queen, where only the final square that the checker is moved over is considered *used*. Vasya on his turn makes a double move as a chess king, where both squares moved over are considered *used*. The checker cannot be moved over a *used* square. The initial square is also considered *used*. The player who cannot make a move loses. Who of the boys can play so that he will win for sure, no matter how his opponent moves? (5 points)

5. A convex polyhedron is given with exactly three faces meeting at each vertex. Each face of the polyhedron is coloured red, yellow or blue. The vertices, where the faces of all three colours meet, are called *multicoloured*. Prove that the number of multicoloured vertices is even. (5 points)

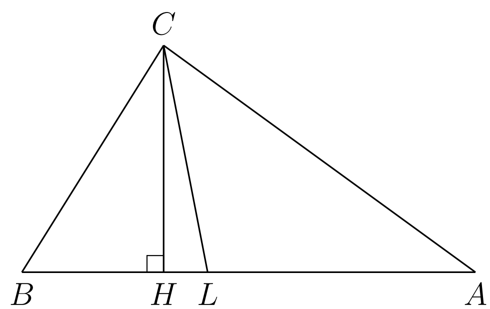
O Level Senior Paper Solutions

Edited by Oleksiy Yevdokimov and Greg Gamble

1. **Solution 1.** No, it is not possible. Indeed, let CL be an angle bisector and CH be an altitude of triangle ABC. Without loss of generality, assume that $\angle A < \angle B$. Then, $BC < CA$. Since point H is located on side AB, $\angle B$ is acute. Since CL is an angle bisector and $\angle BCH = 90^\circ - \angle B < 90^\circ - \angle A = \angle ACH$, point H lies on the line segment BL. Furthermore, since CL is an angle bisector, we have

$$\frac{BL}{LA} = \frac{BC}{CA} < 1 \quad \text{which implies} \quad BL < LA.$$

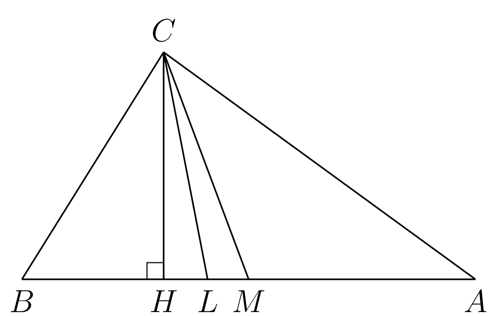
Since $BH + HL = BL$, it follows that $BH + HL < LA$, so that BH, HL and LA do not satisfy the Triangle Inequality. Thus, it is not possible to construct a new triangle from those three parts.



Solution 2. No, it is not possible. Indeed, let CM be a median of triangle ABC. Since for an altitude, angle bisector and median emanating from the same vertex of a triangle, the angle bisector is located between the median and altitude, we get

$$LA > MA = BM > BH + HL$$

which means that the Triangle Inequality is not satisfied by the line segments BH, HL and LA. So, it is not possible to construct a new triangle from those three parts.



2. Let the four numbers be a, b, c and d. We claim that each prime number p has an even exponent in the prime factorisation of the product $abcd$. Indeed, suppose p has exponents α, β, γ and δ in the respective prime factorisations of a, b, c and d. Without loss of generality, assume $\alpha \geq \beta \geq \gamma \geq \delta$. Since d is divisible by

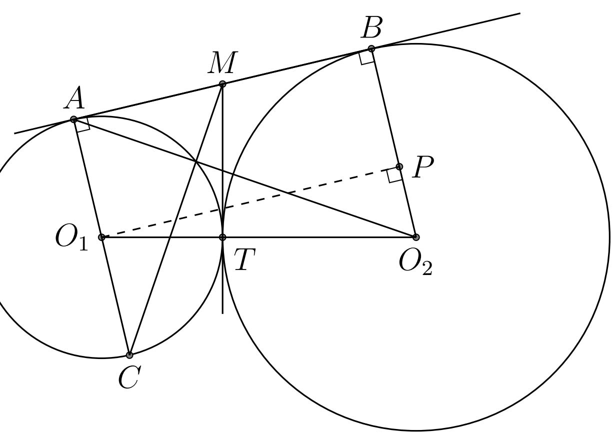
$\gcd(a, b, c)$, we have $\delta \geq \min\{\alpha, \beta, \gamma\} = \gamma$. Hence, $\gamma = \delta$. Since $\operatorname{lcm}(b, c, d)$ is divisible by a, we have $\beta = \max\{\beta, \gamma, \delta\} \geq \alpha$. Hence, $\alpha = \beta$.

Thus, each prime number p has exponent $2\alpha + 2\delta = 2(\alpha + \delta)$ in the prime factorisation of the product $abcd$. As a consequence, $abcd$ is a perfect square.

3. **Solution 1.** Let r_1 and r_2 be the radii of the circles Γ_1 and Γ_2, respectively. Let P be the foot of the perpendicular dropped from point O_1 onto BO_2. Since $O_1O_2 = r_1 + r_2$ and $PO_2 = r_2 - r_1$ in right-angled triangle O_1PO_2, by Pythagoras' Theorem we have

$$O_1P = \sqrt{(r_1 + r_2)^2 - (r_2 - r_1)^2} = 2\sqrt{r_1 r_2}.$$

Since $ABPO_1$ is a rectangle, $AB = O_1P = 2\sqrt{r_1 r_2}$.



Since AB and MT are tangents to both Γ_1 and Γ_2, $MA = MT = MB$, and since $MA + MB = AB$,

$$MA = MB = MT = \tfrac{1}{2}AB = \sqrt{r_1 r_2}.$$

Thus,

$$\frac{CA}{MA} = \frac{2r_1}{\sqrt{r_1 r_2}} = \frac{2\sqrt{r_1 r_2}}{r_2} = \frac{AB}{BO_2}$$

and hence right-angled triangles CAM and ABO_2 are similar.

Since AC and AB are perpendicular, AM and BO_2 are also perpendicular, and, therefore, hypotenuses CM and AO_2 of the two similar triangles are perpendicular to each other. The proof is complete.

Solution 2. Let X be the point of intersection of the line segment AO_2 with Γ_1. Then, $\angle AXC = 90^\circ$ as subtended by diameter of Γ_1. Thus, it is sufficient to prove that points C, X and M lie on a straight line, which is equivalent to showing that $\angle MXO_2 = 90^\circ$.

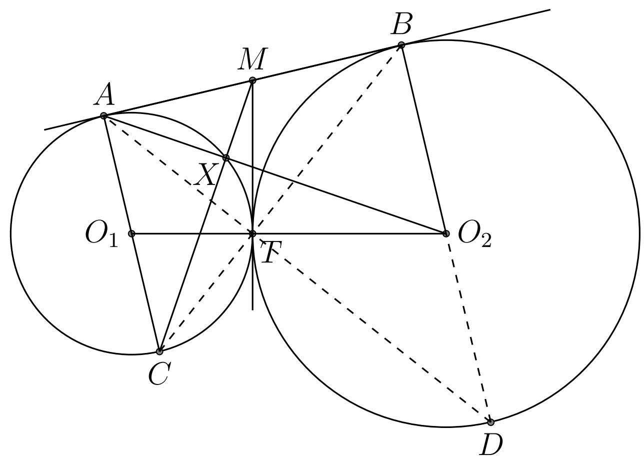
Since $\angle CTA$ is subtended at circumference of O_1 by its diameter, and $MA = MB = MT$ being equal line segments of tangents from the same point imply that AB is the hypotenuse of the circumcircle of ATB,

$$\angle CTA = \angle ATB = 90^\circ.$$

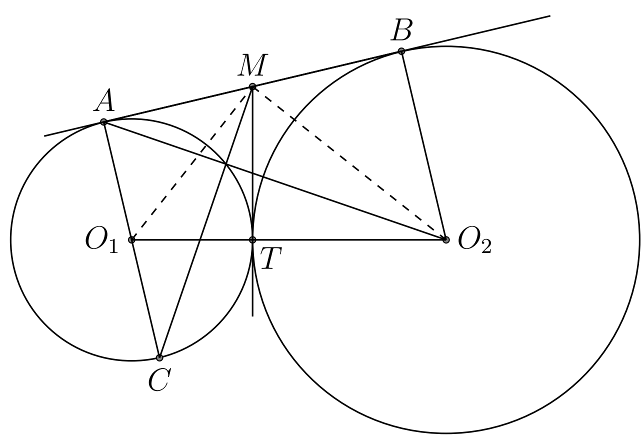
Hence $\angle CTA + \angle ATB = 180°$ and so C, T and B are collinear. Thus,

$$\begin{aligned}\angle TXO_2 &= \angle TCA, \text{ since } AXTC \text{ is cyclic}\\ &= \angle TBO_2 \text{ (alternating angles, since } AC \parallel BO_2)\end{aligned}$$

Therefore, $TXBO_2$ is cyclic. Also, since $\angle MTO_2$ and MBO_2 are right angles, they are supplementary, and hence $MTBO_2$ is cyclic and consequently T, X, B, O_2 and M are concyclic. Therefore, $\angle MXO_2 = \angle MTO_2 = 90°$ and we are done.



Solution 3. Since $AMTO_1$ and $BMTO_2$ are kites, MO_1 and MO_2 are angle bisectors of triangles AMT and BMT respectively, which means $\angle O_1MO_2 = 90°$. Thus, $\angle AMO_1 = \angle BO_2M$ with right-angled triangles AMO_1 and BO_2M being similar. Therefore, there exists a mapping transforming triangle AMO_1 into BO_2M that includes rotation onto 90°, parallel move and dilation. Thus, O_1 as the midpoint of AC moves into M and M as the midpoint of AB moves into O_2 which also means that C moves into A. So a line segment CM moves into AO_2 with the angle between them being equal to the angle of the rotation, which is 90°. This completes the proof.



4. Vasya can win for sure, no matter how his opponent moves. An 8×8 chessboard without the corner square can be divided into 3-square corners – to do so we place one corner around the absent corner square with all other squares to be covered

by ten 3×2 rectangles, where each rectangle is formed by two 3-square corners. Vasya's strategy is to make both moves within a 3-square corner where Petya has moved to. Then, every time before Petya's move each 3-square corner is either entirely open for moves or already closed. Thus, Vasya can always make his two moves. Since, the game is finite, Vasya will win for sure.

Note. A square board of the size $2^n \times 2^n$ without the corner square can be divided into 3-square corners. This can be proven by induction. The problem above is a particular case of $n = 3$ with no need to prove the general case, if a construction for the 8×8 board is provided.

5. **Solution 1.** We call an edge *red-yellow* if it borders both a red face and a yellow face. Consider a multicoloured vertex u. There is exactly one red-yellow edge (call it e_1) emanating from u. Let the other end point of e_1 be v. At v, (the same) red and yellow faces meet. If the third face incident with v is blue, then v is also multicoloured, and no further red-yellow edges emanate from v. Otherwise, another red-yellow edge (call it e_2) emanates from v. Now move along e_2 and repeat the process. Continuing in this way, sooner or later we come to a multicoloured vertex since we cannot return to the vertices we have already passed. Thus, all multicoloured vertices can be divided into pairs as the ends of red-yellow edges.

   Solution 2. We claim that with re-colouring of faces the parity of the quantity of multicoloured vertices does not change. If so, a convex polyhedron can be made of one colour and the statement of the problem is then obviously true. Indeed, let us re-colour some red face yellow. Then, for the two other faces incident with the re-coloured face at a vertex we match such a pair of the two faces with that vertex. The vertex will change its multicolourness, from yes to no or vice versa, if and only if such a vertex matches to a pair of faces which are blue and non-blue. Since there are an even number of such pairs, we conclude that the number of multicoloured vertices is even.